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Relativistic wave equations for spin-2 particles with unique mass

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1. Introduction and summary

It is well known that elementary particles of arbitrary spin can be described by relativistic wave equations of the Dirac form (2.1). Such an equation has the advantage of being derivable from a variational principle without use of subsidiary conditions. If the equation is to be covariant under proper Lorentz transformations, the irreducible set of the matrix coefficients β_k and the generators q_{kl} of the transformation must satisfy the equations (2.7). These equations, as well as the integrability condition (2.6), can be satisfied by different assumptions for the dependence of the q_{kl}'s upon the β_k's [1]. The simplest of these dependences is (2.10), to which we will adhere in this paper, having no physical reasons for more complicated assumptions.

If (2.1) is required to represent a particle of unique mass, all components of the wave function ψ must satisfy Klein–Gordon equations with the same mass constant. Now, with the assumption (2.10), it is impossible to satisfy the condition of unique mass for any spin higher than 1, if the mass term λ in (2.1) is a constant times the unit matrix [2]. In fact, the components of ψ then satisfy a differential equation of higher order than 2, such that the rest mass of the particle can have different values with certain simple ratios. For instance, in the case of spin 2 there are two rest mass values with ratio 1/2. However, a constant times the unit matrix is not the most general form of the mass term. For example, already in the case of spin 1 this assumption is too simple to give Maxwell's equations [3]. In fact, the mass term has only to commute with the generators, (2.8), and satisfy a Hermiticity condition (3.8). There is thus a possibility of obtaining a more general mass spectrum.

In this paper we investigate in some detail the irreducible spin-2 case, where the 14 components of ψ consist of the components of a symmetric tensor of second rank and of a vector. This case of higher spin seems at present to be most interesting physically, due to its bearing on the gravitational problem. Explicit representations, (2.21), (3.10), of the 14-row matrices β_k and the metric form β are deduced by means of the subspace method of Klein [4, 5]. By means of this representation the most general form of the mass term is proved to be an arbitrary linear combination, (4.29), of three commuting projection operators of sum 1. The wave equation is given in component form, (5.1), (5.2). It is proved that all components of ψ satisfy the same fourth order differential equation (5.7), showing that the two rest mass values can be given arbitrary ratio. This ratio can be chosen as 1, if we make the particular assumption (5.11). All components of ψ then satisfy the same Klein–Gordon equation (5.12), showing that in this case the particle has unique mass. On the other hand, with the particular choice (5.13) the differential equation of fourth order degenerates into an ordinary Klein–Gordon equation, giving another case of unique mass. This second case is proved to have positive definite energy (6.6) and can thus be quantized in the conventional way.

The authors intend to generalize the present spin-2 equation to general relativity with a method analogous to that used for the spin-1 case [6]. It may then be possible to obtain the general equations of the gravitational field in empty space by postulating a suitable connection between the symmetric field tensor and the metric fundamental tensor.

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2. Representation of the matrices

We consider relativistic wave equations of the form (Latin indices running from 1 to 4 and $x_4 = i c t$):

$$(\beta^k p_k - \lambda)\,\psi = 0, \tag{2.1}$$

$$\tilde{\psi}\,(\beta^k p_k - \lambda) = 0, \tag{2.2}$$

where

$$p_k = -i\hbar\,\partial_k = -i\hbar \frac{\partial}{\partial x^k}, \tag{2.3}$$

when operating on ψ and with the reversed sign when operating on $\tilde{\psi}$.

Consider a Lorentz transformation, given by the equations

$$p'_k = p_k + \varepsilon_k{}^l p_l, \quad (\varepsilon_{kl} = -\varepsilon_{lk}), \tag{2.4}$$

$$\psi' = Q\,\psi = (1 + \tfrac{1}{2}\varepsilon^{rs} q_{rs})\,\psi, \quad (q_{rs} = -q_{sr}), \tag{2.5}$$

$$[q_{kl}, q_{rs}] = \delta_{lr}\, q_{ks} - \delta_{kr}\, q_{ls} + \delta_{ls}\, q_{rk} - \delta_{ks}\, q_{rl}. \tag{2.6}$$

The covariance of (2.1) under such a transformation implies the relations

$$[q_{kl}, \beta_r] = \delta_{lr}\,\beta_k - \delta_{kr}\,\beta_l, \tag{2.7}$$

$$[\lambda, Q] = 0 \quad \text{or} \quad [\lambda, q_{kl}] = 0. \tag{2.8}$$

In this paper we investigate the irreducible 14-row representation of the β_k's, corresponding to a wave function, consisting of a symmetric tensor of second rank and of a vector.

According to Klein [4] we consider the 4-dimensional space as a subspace of a 5-dimensional one. The $q_{\varrho\sigma}$'s (Greek indices running from 1 to 5) for the Lorentz transformations of the 5-dimensional space satisfy (2.6) with Greek indices. If we put

$$\beta_\varrho = i\, q_{\varrho 5}, \quad (\beta_5 = 0), \tag{2.9}$$

(2.6) with Greek indices contains (2.6) and (2.7) with Latin indices and the following special dependence of the q_{kl}'s upon the β_k's:

$$q_{kl} = [\beta_k, \beta_l]. \tag{2.10}$$

(2.10) and (2.7) give:

$$[[\beta_k, \beta_l], \beta_r] = \delta_{lr}\, \beta_k - \delta_{kr}\, \beta_l. \tag{2.11}$$

To obtain a representation of the β_k's of our case, we make use of a method, developed by Klein and worked out in some detail by Hjalmars [5] for the Petiau–Duffin–Kemmer β_k-matrices. By this method we construct an explicit form of the $q_{\varrho\sigma}$'s of the irreducible representation of the 5-dimensional Lorentz group, where the 14 components of the operand consist of the elements of a 5-dimensional symmetric tensor $\psi_{\alpha\beta}$ with $\psi_\alpha{}^\alpha = 0$. Under the restriction to 4-dimensional transformations this 5-dimensional tensor breaks up, as required, in a symmetric 4-dimensional tensor $\psi_{kl} = \psi_{lk}$, a scalar $\psi_{55} = -\psi_k{}^k$ and a 4-vector ψ_{5l}.

The representation is obtained by identifying (2.5) with Greek indices with the explicit 5-dimensional Lorentz transformation of $\psi_{\alpha\beta}$. We thus obtain:

$$\psi_{\alpha\beta} + \tfrac{1}{2}\, \varepsilon^{\varrho\sigma} (q_{\varrho\sigma}\, \psi)_{\alpha\beta} = \psi_{\alpha\beta} + \varepsilon_\alpha{}^\varrho\, \psi_{\varrho\beta} + \varepsilon_\beta{}^\varrho\, \psi_{\alpha\varrho}. \tag{2.12}$$

The arbitrariness of the $\varepsilon_{\alpha\beta}$'s gives:

$$(q_{\varrho\sigma}\, \psi)_{\alpha\beta} = \delta_{\alpha\varrho}\, \psi_{\sigma\beta} - \delta_{\alpha\sigma}\, \psi_{\varrho\beta} + \delta_{\varrho\beta}\, \psi_{\alpha\sigma} - \delta_{\sigma\beta}\, \psi_{\alpha\varrho}, \tag{2.13}$$

which can be written

$$\langle{}_{\alpha\beta} | q_{\varrho\sigma} |{}^{\alpha'\beta'}\rangle\, \psi_{\alpha'\beta'} = \tfrac{1}{2} [\delta_{\alpha\varrho}\, \delta_\sigma{}^{\alpha'}\, \delta_\beta{}^{\beta'} - \mathrm{ex}\,(\varrho\,\sigma) + \mathrm{ex}\,(\alpha\,\beta) + \mathrm{ex}\,(\alpha'\,\beta')]\, \psi_{\alpha'\beta'}, \tag{2.14}$$

where $\mathrm{ex}\,(\varrho\,\sigma)$ means the entire preceding expression with ϱ and σ exchanged.

Now, we are only considering operands ψ's and $\tilde{\psi}$'s, fulfilling

$$\psi_{\alpha\beta} = \psi_{\beta\alpha}, \qquad \tilde{\psi}_{\alpha\beta} = \tilde{\psi}_{\beta\alpha}, \tag{2.15}$$

$$\psi_\alpha{}^\alpha = 0, \qquad \tilde{\psi}_\alpha{}^\alpha = 0, \tag{2.16}$$

i.e. being of the form

$$\psi_{\alpha\beta} \quad \text{or} \quad \tilde{\psi}_{\alpha\beta} = \tfrac{1}{2} (A_{\alpha\beta} + A_{\beta\alpha}) - \tfrac{1}{5} A_{\alpha'}{}^{\alpha'}\, \delta_{\alpha\beta}. \tag{2.17}$$

Thus, every operator, operating on a ψ or a $\tilde{\psi}$, must have the property of giving a new operand, fulfilling the same conditions. The matrix elements of every operator must consequently satisfy

$$\langle {}_{\alpha\beta} | \mathrm{Op} | {}^{\alpha'\beta'} \rangle \, \psi_{\alpha'\beta'} = \langle {}_{\beta\alpha} | \mathrm{Op} | {}^{\alpha'\beta'} \rangle \, \psi_{\alpha'\beta'}, \tag{2.18}$$

$$\langle {}_{\alpha}{}^{\alpha} | \mathrm{Op} | {}^{\alpha'\beta'} \rangle \, \psi_{\alpha'\beta'} = 0, \tag{2.19}$$

and correspondingly for $\tilde{\psi}$. (2.18) and (2.19) must in particular be fulfilled for $q_{\varrho\sigma}$. Since the brackets in (2.14) are easily seen to satisfy (2.18) and (2.19), the relevant part of the matrix elements for $q_{\varrho\sigma}$ must be:

$$\langle {}_{\alpha\beta} | q_{\varrho\sigma} | {}_{\alpha'\beta'} \rangle = \tfrac{1}{2} [\delta_{\alpha\varrho} \, \delta_{\alpha'\sigma} \, \delta_{\beta\beta'} - \mathrm{ex}\,(\varrho\,\sigma) + \mathrm{ex}\,(\alpha\beta) + \mathrm{ex}\,(\alpha'\beta')]. \tag{2.20}$$

By means of (2.9) we obtain

$$\langle {}_{\alpha\beta} | \beta_{\varrho} | {}_{\alpha'\beta'} \rangle = \frac{i}{2} [\delta_{\alpha\varrho} \, \delta_{\alpha'5} \, \delta_{\beta\beta'} - \delta_{\alpha 5} \, \delta_{\alpha'\varrho} \, \delta_{\beta\beta'} + \mathrm{ex}\,(\alpha\beta) + \mathrm{ex}\,(\alpha'\beta')]. \tag{2.21}$$

In this representation the relevant part of the unit matrix is

$$\langle {}_{\alpha\beta} | 1 | {}_{\alpha'\beta'} \rangle = \tfrac{1}{2} [\tfrac{1}{2} \delta_{\alpha\alpha'} \, \delta_{\beta\beta'} - \tfrac{1}{10} \delta_{\alpha\beta} \, \delta_{\alpha'\beta'} + \mathrm{ex}\,(\alpha\beta) + \mathrm{ex}\,(\alpha'\beta')], \tag{2.22}$$

since this matrix is seen to reproduce a ψ or a $\tilde{\psi}$ with the property (2.17).

Taking the complex conjugate of (2.14), it is seen that the $q_{\varrho\sigma}$'s are anti-Hermitian. The β_{ϱ}'s are thus Hermitian by (2.9).

Since our procedure starts from an irreducible representation of the 5-dimensional group, the representation (2.20) of $q_{\varrho\sigma}$ is automatically irreducible. Since the q_{kl}'s of (2.20) can be obtained from the q_{k5}'s by means of (2.9), (2.10), the set β_k of (2.21) must be irreducible. Of course, the set q_{kl} is not irreducible, but contains three irreducible representations of the 4-dimensional Lorentz group, corresponding to the transformation of $\psi_{kl} - \delta_{kl} \psi_r{}^r$, $\psi_r{}^r$ and ψ_{5k} respectively.

3. Minimal equation and metric form

In order to determine the minimal equation we calculate $q_{\varrho\sigma}{}^2$. We obtain by means of (2.20):

$$\begin{aligned}
&\langle {}_{\alpha\beta} | q_{\varrho\sigma}{}^2 | {}_{\alpha'\beta'} \rangle \\
&\quad = \langle {}_{\alpha\beta} | q_{\varrho\sigma} | {}^{\alpha''\beta''} \rangle \langle {}_{\alpha''\beta''} | q_{\varrho\sigma} | {}_{\alpha'\beta'} \rangle \\
&\quad = \tfrac{1}{2} (\delta_{\alpha\varrho} \, \delta_{\sigma}{}^{\alpha''} \, \delta_{\beta}{}^{\beta''} - \delta_{\alpha\sigma} \, \delta_{\varrho}{}^{\alpha''} \, \delta_{\beta}{}^{\beta''} + \delta_{\alpha\varrho} \, \delta_{\sigma}{}^{\beta''} \, \delta_{\beta}{}^{\alpha''} - \delta_{\alpha\sigma} \, \delta_{\varrho}{}^{\beta''} \, \delta_{\beta}{}^{\alpha''}) \times \\
&\qquad\qquad \times (\delta_{\alpha''\varrho} \, \delta_{\alpha'\sigma} \, \delta_{\beta''\beta'} - \delta_{\alpha''\sigma} \, \delta_{\alpha'\varrho} \, \delta_{\beta''\beta'}) + \mathrm{ex}\,(\alpha\beta) + \mathrm{ex}\,(\alpha'\beta') \\
&\quad = \tfrac{1}{2} (\delta_{\alpha\varrho} \, \delta_{\sigma\varrho} \, \delta_{\beta\beta'} \, \delta_{\alpha'\sigma} - \delta_{\alpha\varrho} \, \delta_{\beta\beta'} \, \delta_{\alpha'\varrho} - \\
&\qquad\qquad - \delta_{\alpha\sigma} \, \delta_{\beta\beta'} \, \delta_{\alpha'\sigma} + \delta_{\alpha\sigma} \, \delta_{\varrho\sigma} \, \delta_{\beta\beta'} \, \delta_{\alpha'\varrho} + \delta_{\alpha\varrho} \, \delta_{\beta'\sigma} \, \delta_{\beta\varrho} \, \delta_{\alpha'\sigma} - \\
&\qquad\qquad - \delta_{\alpha\varrho} \, \delta_{\beta'\sigma} \, \delta_{\beta\sigma} \, \delta_{\alpha'\varrho} - \delta_{\alpha\sigma} \, \delta_{\beta'\varrho} \, \delta_{\beta\varrho} \, \delta_{\alpha'\sigma} + \delta_{\alpha\sigma} \, \delta_{\beta'\varrho} \, \delta_{\beta\sigma} \, \delta_{\alpha'\varrho}) + \mathrm{ex}\,(\alpha\beta) + \mathrm{ex}\,(\alpha'\beta'),
\end{aligned} \tag{3.1}$$

$$\langle_{\alpha\beta}| q_{\varrho\sigma}{}^2 |_{\alpha'\beta'}\rangle$$
$$= \tfrac{1}{2}[\delta_{\alpha\varrho}\,\delta_{\alpha'\sigma}\,\delta_{\beta\beta'}\,\delta_{\sigma\varrho} - \delta_{\alpha\varrho}\,\delta_{\alpha'\varrho}\,\delta_{\beta\beta'} + \delta_{\alpha\varrho}\,\delta_{\alpha'\sigma}\,\delta_{\beta\varrho}\,\delta_{\beta'\sigma} - \delta_{\alpha\varrho}\,\delta_{\alpha'\varrho}\,\delta_{\beta\sigma}\,\delta_{\beta'\sigma}] +$$
$$+ \mathrm{ex}\,(\varrho\,\sigma) + \mathrm{ex}\,(\alpha\,\beta) + \mathrm{ex}\,(\alpha'\,\beta'). \quad (3.2)$$

If we calculate the higher powers of $q_{\varrho\sigma}$ in the same way, we find that $q_{\varrho\sigma}$ satisfies

$$q_{\varrho\sigma}{}^5 + 5\,q_{\varrho\sigma}{}^3 + 4\,q_{\varrho\sigma} = 0 \quad (3.3)$$

with the roots 0, $\pm i$, $\pm 2i$. This is the minimal equation for $q_{\varrho\sigma}$, which can be proved as follows. Since $q_{\varrho\sigma}$ is an anti-Hermitian matrix with only real elements, its minimal equation can contain either only even terms or only odd terms. The constant term of the minimal equation must be zero, since the unit matrix (2.22) contains matrix elements, which cannot occur in the dignities of $q_{\varrho\sigma}$. Thus the minimal equation cannot be even. It is easily seen that the possible equations of first and third degree are not satisfied by $q_{\varrho\sigma}$. (3.3) is thus the minimal equation. The eigenvalues of $i\,q_{\varrho\sigma}$ are consequently 0, ± 1, ± 2. By (2.9) this is in particular the case for β_k, having the minimal equation

$$\beta_k^5 - 5\,\beta_k^3 + 4\,\beta_k = 0. \quad (3.4)$$

It is now clear that all assumptions, made by Madhava Rao [8] for calculating the commutation rules for the matrices of spin 2, are satisfied by our q_{kl}, β_k. These matrices are thus a special, irreducible representation of the spin-2 algebra of Madhava Rao.

As well known, a Hermitian matrix β, fulfilling

$$\beta^2 = 1, \quad (3.5)$$

$$\beta\,\beta_k + \beta_k\,\beta = 0 \quad (k = 1, 2, 3), \quad (3.6)$$

$$[\beta, \beta_4] = 0, \quad (3.7)$$

$$\lambda^+ = -\beta\,\lambda\,\beta, \quad (3.8)$$

is a satisfactory metric form, which can be used in the relation

$$\tilde{\psi} = \psi^*\,\beta. \quad (3.9)$$

We will prove for our case that the Hermitian matrix

$$\beta = \tfrac{1}{3}(2\,\beta_4^4 - 8\,\beta_4^2 + 3) \quad (3.10)$$

can be used as a metric form. We obtain by (3.4):

$$\beta^2 = \tfrac{1}{9}(4\,\beta_4^8 - 32\,\beta_4^6 + 76\,\beta_4^4 - 48\,\beta_4^2 + 9) = 1, \quad (3.11)$$

which proves (3.5). Furthermore, if we make use of equation (29) of Madhava Rao [8], i.e.

$$\beta_4^4 \beta_k + \beta_k \beta_4^4 = \beta_4^2 \beta_k + 6 \beta_4 \beta_k \beta_4 + \beta_k \beta_4^2, \tag{3.12}$$

and of (2.11), we obtain ($k \neq 4$):

$$\beta \beta_k + \beta_k \beta = 2 (2 \beta_4 \beta_k \beta_4 - \beta_4^2 \beta_k - \beta_k \beta_4^2 + \beta_k) = 0, \tag{3.13}$$

which proves (3.6). (3.7) is evidently fulfilled. (3.8) will be proved in the next section.

The matrix elements of β can be calculated by means of (3.10), (2.22), (3.2) and the corresponding equation for $q_{\varrho\sigma}{}^4$. When inserting the result in (3.9), we obtain the following connection between the components of $\tilde{\psi}$ and ψ^*:

$$\tilde{\psi}_{\alpha\beta} = \pm \psi^*{}_{\alpha\beta}, \tag{3.14}$$

where the minus sign is to be chosen, if one of the indices α, β is 4 or 5 and the other 1, 2 or 3, whereas the plus sign is to be chosen in all other cases.

4. The mass term

As well known, (2.7) means that the Q-transformation of β_k is identical with the transformation of β_k as a vector component. On the other hand, (2.8) means that λ must be an invariant under the Q-transformation. If λ is expressed as a function of the irreducible β_k's, this expression must thus have the form of an invariant formed by the β_k's as vector components. The most general form of λ is consequently

$$\lambda = \dots \beta_r \dots \beta^r \dots , \tag{4.1}$$

with all indices contracted.

From (4.1), (3.6) and (3.7) follows immediately that λ and β commute. The condition (3.8) is thus equivalent to the condition that λ is anti-Hermitian:

$$\lambda^+ = -\lambda. \tag{4.2}$$

In order to reduce (4.1) to its simplest form, we start with the investigation of the Hermitian invariant

$$B = \beta_k \beta^k = \beta_\varrho \beta^\varrho. \tag{4.3}$$

By putting $\sigma = 5$ and summing over ϱ in (3.2), we obtain

$$\langle_{\alpha\beta} | B |_{\alpha'\beta'}\rangle = \tfrac{1}{2} [\delta_{\alpha\alpha'} \delta_{\beta\beta'} + 5 \delta_{\alpha 5} \delta_{\alpha' 5} \delta_{\beta\beta'} - \delta_{\alpha\beta} \delta_{\alpha' 5} \delta_{\beta' 5} - \delta_{\alpha 5} \delta_{\alpha'\beta'} \delta_{\beta 5}] + \mathrm{ex}\,(\alpha\beta) + \mathrm{ex}\,(\alpha'\beta'). \tag{4.4}$$

After calculating the higher powers of B, we obtain the minimal equation

$$B^3 - 19\,B^2 + 104\,B - 140 = 0 \tag{4.5}$$

with the roots $B_1 = 2$, $B_2 = 10$, $B_3 = 7$.

Any polynomial of B can now be written as a linear polynomial of three projection operators ω_1, ω_2 and ω_3, satisfying:

$$\omega_k\,\omega_l = \omega_k\,\delta_{kl}, \tag{4.6}$$

$$\omega_1 + \omega_2 + \omega_3 = 1, \tag{4.7}$$

$$B = 2\,\omega_1 + 10\,\omega_2 + 7\,\omega_3. \tag{4.8}$$

The idempotents ω_1, ω_2, ω_3 are seen to be:

$$\omega_1 = \tfrac{1}{40}\,(B^2 - 17\,B + 70), \tag{4.9}$$

$$\omega_2 = \tfrac{1}{24}\,(B^2 - 9\,B + 14), \tag{4.10}$$

$$\omega_3 = -\tfrac{1}{15}\,(B^2 - 12\,B + 20). \tag{4.11}$$

By means of (4.4) we obtain:

$$\begin{aligned}\langle_{\alpha\beta}|\,\omega_1\,|_{\alpha'\beta'}\rangle &= \tfrac{1}{2}\,[\tfrac{1}{2}\,\delta_{\alpha\alpha'}\,\delta_{\beta\beta'} - \tfrac{1}{8}\,\delta_{\alpha\beta}\,\delta_{\alpha'\beta'} - \delta_{\alpha\alpha'}\,\delta_{\beta 5}\,\delta_{\beta' 5} + \tfrac{1}{8}\,\delta_{\alpha'\beta'}\,\delta_{\alpha 5}\,\delta_{\beta 5} + \tfrac{1}{8}\,\delta_{\alpha\beta}\,\delta_{\alpha' 5}\,\delta_{\beta' 5} + \\ &\quad + \tfrac{3}{8}\,\delta_{\alpha 5}\,\delta_{\beta 5}\,\delta_{\alpha' 5}\,\delta_{\beta' 5}] + \text{ex}\,(\alpha\,\beta) + \text{ex}\,(\alpha'\,\beta'),\end{aligned} \tag{4.12}$$

$$\begin{aligned}\langle_{\alpha\beta}|\,\omega_2\,|_{\alpha'\beta'}\rangle &= \tfrac{1}{2}\,[\tfrac{1}{40}\,\delta_{\alpha\beta}\,\delta_{\alpha'\beta'} - \tfrac{1}{8}\,\delta_{\alpha'\beta'}\,\delta_{\alpha 5}\,\delta_{\beta 5} - \tfrac{1}{8}\,\delta_{\alpha\beta}\,\delta_{\alpha' 5}\,\delta_{\beta' 5} + \tfrac{5}{8}\,\delta_{\alpha 5}\,\delta_{\beta 5}\,\delta_{\alpha' 5}\,\delta_{\beta' 5}] + \\ &\quad + \text{ex}\,(\alpha\,\beta) + \text{ex}\,(\alpha'\,\beta'),\end{aligned} \tag{4.13}$$

$$\begin{aligned}\langle_{\alpha\beta}|\,\omega_3\,|_{\alpha'\beta'}\rangle &= \tfrac{1}{2}\,[\delta_{\alpha\alpha'}\,\delta_{\beta 5}\,\delta_{\beta' 5} - \delta_{\alpha 5}\,\delta_{\beta 5}\,\delta_{\alpha' 5}\,\delta_{\beta' 5}] + \text{ex}\,(\alpha\,\beta) + \text{ex}\,(\alpha'\,\beta').\end{aligned} \tag{4.14}$$

The components of $\omega_k\,\psi$ are

$$(\omega_1\,\psi)_{kl} = \psi_{kl} - \tfrac{1}{4}\,\psi_r{}^r\,\delta_{kl}, \tag{4.15}$$

$$(\omega_2\,\psi)_{kk} = \tfrac{1}{4}\,\psi_r{}^r, \tag{4.16}$$

$$(\omega_2\,\psi)_{55} = \psi_{55} = -\,\psi_r{}^r, \tag{4.17}$$

$$(\omega_3\,\psi)_{5l} = (\omega_3\,\psi)_{l5} = \psi_{5l}, \tag{4.18}$$

all other components being 0. The equations (4.15)–(4.18) show that the operators ω_1, ω_2 and ω_3 project into the three subspaces, spanned by the traceless tensor $\psi_{kl} - \frac{1}{4}\psi_r{}^r\,\delta_{kl}$, the trace $\psi_r{}^r$ and the vector ψ_{5k} respectively.

By (2.21) and (4.14) we obtain:

$$\omega_3\,\beta_k + \beta_k\,\omega_3 = \beta_k, \tag{4.19}$$

giving by (4.6) and (4.7):

$$\omega_3\,\beta_k = \beta_k\,(\omega_1 + \omega_2), \qquad (\omega_1 + \omega_2)\,\beta_k = \beta_k\,\omega_3, \tag{4.20}$$

$$\omega_1\,\beta_k\,\omega_1 = \omega_2\,\beta_k\,\omega_2 = \omega_3\,\beta_k\,\omega_3 = 0, \tag{4.21}$$

$$\omega_1\,\beta_k\,\omega_2 = \omega_2\,\beta_k\,\omega_1 = 0, \tag{4.22}$$

$$\omega_3\,\beta_k\,\omega_1 = \beta_k\,\omega_1, \qquad \omega_1\,\beta_k\,\omega_3 = \omega_1\,\beta_k, \tag{4.23}$$

$$\omega_3\,\beta_k\,\omega_2 = \beta_k\,\omega_2, \qquad \omega_2\,\beta_k\,\omega_3 = \omega_2\,\beta_k, \tag{4.24}$$

$$\omega_3\,\beta_k\,\beta_l = \beta_k\,\beta_l\,\omega_3. \tag{4.25}$$

By (2.21) and (4.13) we get:

$$\omega_2\,\beta_k\,\beta_l = \omega_2\,\beta_l\,\beta_k, \qquad \beta_k\,\beta_l\,\omega_2 = \beta_l\,\beta_k\,\omega_2, \tag{4.26}$$

$$\beta_k\,\omega_2\,\beta^k = \tfrac{5}{2}\,\omega_3. \tag{4.27}$$

We are now able to prove that every expression of the type (4.1) can be reduced to a polynomial in B or equivalently a linear polynomial in ω_1, ω_2, ω_3.

We first show that any expression of the type (4.1) can be written as a linear polynomial in the ω_r's times an expression of the type (4.1), containing at least two β_k's less than the original expression. Denoting by [...] all expressions of this reduced type, we obtain by (2.11), (4.3), (4.8), (4.20), (4.26) and (4.27):

$$\begin{aligned}
&\beta_k \ldots \beta_r \ldots \beta^r \ldots \beta^k \ldots \\
&\quad = \beta_k\,\beta_l\,\beta^k \cdots + \beta_k\,\beta^k \cdots + [\cdots] \\
&\quad = \tfrac{1}{2}\,\beta_l\,\beta_k\,\beta^k \cdots + \tfrac{1}{2}\,\beta_k\,\beta^k\,\beta_l \cdots + [\cdots] \\
&\quad = \tfrac{1}{2}\,\beta_l\,\{2\,(\omega_1 + \omega_2) + 8\,\omega_2 + 7\,\omega_3\} \cdots + [\cdots] \\
&\quad = \omega_3\,\beta_l \cdots + 4\,\beta_l\,\omega_2 \cdots + \tfrac{7}{2}\,(\omega_1 + \omega_2)\,\beta_l \cdots + [\cdots] \\
&\quad = 4\,\beta_l\,\omega_2 \cdots + [\cdots] = 4\,\beta_l\,\omega_2\,\beta_m\,\beta^l \cdots + 4\,\beta_l\,\omega_2\,\beta^l \cdots + [\cdots] \\
&\quad = 4\,\beta_l\,\omega_2\,\beta^l\,\beta_m \cdots + [\cdots] = [\cdots].
\end{aligned} \tag{4.28}$$

Diminishing in this way successively the number of β_k's in the expression [...], we obtain at last an expression, not containing any β_k's at all. The most general form of λ can thus be written:

$$\lambda = -i\hbar\,(\varkappa_1\,\omega_1 + \varkappa_2\,\omega_2 + \varkappa_3\,\omega_3), \tag{4.29}$$

where $\varkappa_1$, $\varkappa_2$ and $\varkappa_3$ are arbitrary constant c-numbers, which by (4.2) must be real.

5. The wave equation in component form

If we insert (4.29), (2.21), (4.12), (4.13) and (4.14) in (2.1), we obtain:

$$\partial_k\,\psi_{5l} + \partial_l\,\psi_{5k} = -i\,\varkappa_1\,\psi_{kl} - \frac{i}{4}(\varkappa_2 - \varkappa_1)\,\psi_r{}^r\,\delta_{kl}, \tag{5.1}$$

$$\partial_k\,\psi_r{}^r + \partial_r\,\psi_k{}^r = i\,\varkappa_3\,\psi_{5k}, \tag{5.2}$$

$$\partial_k\,\psi_5{}^k = -\frac{i}{2}\varkappa_2\,\psi_k{}^k, \tag{5.3}$$

where the 55-component (5.3) of the wave equation of course also can be obtained from (5.1) by putting $k=l$, and summing over k.

In order to investigate the mass, we will now derive the differential equation of higher order, which is satisfied by *all* components of the wave function of (2.1). If we insert (5.3) in (5.1), we obtain:

$$\varkappa_1\,\varkappa_2\,\psi_{kl} = i\,\varkappa_2\,(\partial_k\,\psi_{5l} + \partial_l\,\psi_{5k}) + \frac{i}{2}(\varkappa_1 - \varkappa_2)\,\partial_r\,\psi_5{}^r\,\delta_{kl}. \tag{5.4}$$

If ψ_{kl} of (5.4) is inserted in (5.2), we get:

$$(5\,\varkappa_1 + \varkappa_2)\,\partial_k\,\partial_r\,\psi_5{}^r + 2\,\varkappa_2\,(\partial_r\,\partial^r - \varkappa_1\,\varkappa_3)\,\psi_{5k} = 0. \tag{5.5}$$

Now, multiply (5.5) by ∂_k and sum over k:

$$[(5\,\varkappa_1 + 3\,\varkappa_2)\,\partial_r\,\partial^r - 2\,\varkappa_1\,\varkappa_2\,\varkappa_3]\,(\partial_k\,\psi_5{}^k) = 0. \tag{5.6}$$

The operator bracket in (5.6), operating on (5.5), gives a fourth order differential equation for ψ_{5k}. If $\varkappa_1 \neq 0$ and $\varkappa_2 \neq 0$, it is seen from (5.4) and (2.16) that this equation is also satisfied by ψ_{kl} and ψ_{55}, and thus by the column ψ. The equation is:

$$[\partial_r\,\partial^r - \varkappa_1\,\varkappa_3]\,[(5\,\varkappa_1 + 3\,\varkappa_2)\,\partial_r\,\partial^r - 2\,\varkappa_1\,\varkappa_2\,\varkappa_3]\,\psi = 0. \tag{5.7}$$

If $5\,\varkappa_1 + 3\,\varkappa_2 \neq 0$, (5.7) can be interpreted as the field equations for an elementary particle with two possible values of the rest mass:

$$m_1 = \frac{\hbar^2}{c} \sqrt{\varkappa_1 \varkappa_3}, \tag{5.8}$$

$$m_2 = \frac{\hbar^2}{c} \sqrt{\frac{2 \varkappa_1 \varkappa_2 \varkappa_3}{5 \varkappa_1 + 3 \varkappa_2}} = \sqrt{\frac{2 \varkappa_2}{5 \varkappa_1 + 3 \varkappa_2}}\, m_1. \tag{5.9}$$

If λ of (4.29) is a constant times the unit matrix, which is the case investigated by Bhabha [2], we have $\varkappa_1 = \varkappa_2 = \varkappa_3$. In this case (5.9) gives:

$$m_2 = \tfrac{1}{2} m_1, \tag{5.10}$$

which is the result of Bhabha.

On the other hand (5.9) shows that in our more general case any ratio between the mass values m_1 and m_2 is possible.

Moreover, (5.7) reveals two possibilities of choosing $\varkappa_1$ and $\varkappa_2$ so as to obtain a unique value of the rest mass:

The first possibility consists in making $m_1 = m_2$. By (5.9) this can be achieved by putting

$$\varkappa_2 = -5 \varkappa_1. \tag{5.11}$$

With this choice (5.4) and (5.5) give:

$$(\partial_r \partial^r - \varkappa_1 \varkappa_3)\, \psi = 0. \tag{5.12}$$

In this case, the wave equation (2.1) can consequently be interpreted as the equation for a particle with the unique mass m_1 of (5.8).

The second possibility consists in choosing

$$\varkappa_2 = -\tfrac{5}{3} \varkappa_1, \tag{5.13}$$

making (5.7) identical with (5.12). In this case (5.6), (5.3), (5.2) and (5.4) give:

$$\partial_k \psi_5{}^k = 0, \tag{5.14}$$

$$\psi_k{}^k = 0, \tag{5.15}$$

$$\partial_r \psi_k{}^r = i \varkappa_3 \psi_{5k}, \tag{5.16}$$

$$\partial_k \partial_r \psi^{kr} = 0, \tag{5.17}$$

$$\psi_{kl} = \frac{i}{\varkappa_1} (\partial_k \psi_{5l} + \partial_l \psi_{5k}). \tag{5.18}$$

6. Energy and spin

For the special case $\varkappa_1 = \varkappa_2 = \varkappa_3$ we can, as shown in [9], use the energy momentum density tensor

$$T_{kl} = -ic\left\{-i\hbar\,\tilde{\psi}\,\beta_l\,(\partial_k\,\psi) - \frac{i\hbar}{2}\,\partial_m\,[-\tilde{\psi}\,\beta^m\,q_{kl}\,\psi + \tilde{\psi}\,\beta_l\,q_k{}^m\,\psi + \tilde{\psi}\,\beta_k\,q_l{}^m\,\psi]\right\}, \quad (6.1)$$

and the spin angular momentum density

$$S_{kl} = -i\hbar\,\tilde{\psi}\,\beta_4\,q_{kl}\,\psi. \quad (6.2)$$

In our more general case, where $\varkappa_1$, $\varkappa_2$ and $\varkappa_3$ are not equal, (6.1) and (6.2) can still be interpreted as energy momentum and spin densities. In fact, for the proof of the relations, necessary for this interpretation, we do not need the assumption that λ is a constant times the unit matrix, but only the condition (2.8), which must be satisfied also in the most general case.

If in (6.1) $k = l = 4$, m runs only from 1 to 3, and, consequently, the relevant part of the energy density is

$$H = -\hbar\,c\,\tilde{\psi}\,\beta_4\,\partial_4\,\psi. \quad (6.3)$$

From (6.2) and (3.3) follows that in our case the highest eigenvalue of the spin is 2. Thus, according to [2], the corresponding particle can be classified as having spin 2.

We will now investigate the positive definiteness of the total energy in the two possible cases of unique rest mass. For this purpose we follow the ordinary procedure of decomposing the field into Fourier waves and investigating every wave, when brought to rest. The wave is then of the form:

$$\psi = \varphi\,\exp\,(\pm i\,c\,\sqrt{\varkappa_1\,\varkappa_3}\,t), \quad (6.4)$$

where φ is a constant column.

If (6.4) is inserted in (5.1), (5.2) and (5.3) with (5.11) or (5.13), all components of φ can be expressed in φ_{5k}. If these expressions are brought into (6.3), we obtain for the case (5.11):

$$H = 4\,\hbar\,c\,\varkappa_3\left[\sum_{n=1}\overset{*}{\varphi}_{5n}\,\varphi_{5n} - \overset{*}{\varphi}_{54}\,\varphi_{54}\right] \quad (6.5)$$

and for the case (5.13):

$$H = 4\,\hbar\,c\,\varkappa_3\sum_{n=1}^{3}\overset{*}{\varphi}_{5n}\,\varphi_{5n}. \quad (6.6)$$

For the case (5.11) of unique mass, (6.5) shows that the energy is not positive definite and a quantization can thus not be made without making use of indefinite metric. In the unique mass case (5.13), on the other hand, we see from (6.6) that the energy is positive definite, so that the quantization can be performed in the conventional way.

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